Eur. Phys. J. C (2015) 75:377
DOI 10.1140/epjc/s10052-015-3579-2

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

The complex-mass scheme and unitarity in perturbative quantum field theory

Ansgar Denner[a], **Jean-Nicolas Lang**[b]

Institut für Theoretische Physik und Astrophysik, Universität Würzburg, 97074 Würzburg, Germany

Received: 22 June 2015 / Accepted: 23 July 2015 / Published online: 21 August 2015

Abstract We investigate unitarity within the complex-mass scheme, a convenient universal scheme for perturbative calculations involving unstable particles in quantum field theory which guarantees exact gauge invariance. Since this scheme requires one to introduce complex masses and complex couplings, the Cutkosky cutting rules, which express perturbative unitarity in theories of stable particles, are no longer valid. We derive corresponding rules for scalar theories with unstable particles based on Veltman's largest-time equation and prove unitarity in this framework.

1 Introduction

With the discovery of the Higgs boson at the large hadron collider nature again reflects not only the relevance of fundamental principles such as gauge invariance as they are incorporated in theories like the standard model (SM), but also that unstable particles are as much important as stable ones. The majority of the known fundamental particles are unstable, and in physical observables unstable particles usually play a significant role.

Precision predictions within perturbative quantum field theories (QFT) are still a challenging task, especially when unstable particles are involved. Unstable particles are of a non-perturbative nature in the sense that in the usual leading-order (LO) perturbation theory all particles are stable. As a consequence near thresholds or resonances observables even diverge in standard perturbation theory because important contributions are missing. A proper treatment requires the inclusion of finite-width effects via a finite imaginary part in the denominator of the Feynman propagator at least near the poles of unstable particles. In perturbation theory, this imaginary part results from a resummation of self-energies.

To date there is no fully established treatment of unstable particles within perturbation theory, although many solutions have been proposed. The problem arises from the need to resum self-energies, thus introducing a mixing of perturbative orders. If done carelessly, this leads to violation of gauge invariance and gauge independence. Thus, the naive modification of the propagator to include a constant fixed width, the so-called fixed-width scheme, violates Ward identities. When performing precision calculations for Z production at LEP it turned out that the renormalization of the Z-boson mass in the usual on-shell renormalization scheme introduces a gauge dependence, as pointed out by Stuart and Sirlin [1–3].

For inclusive observables that are dominated by the production of on-shell unstable particles with a small width, finite-width effects can be neglected if the required precision is small compared to the ratio of width and mass of the unstable particles. This so-called narrow-width approximation is, however, insufficient for many applications. A straightforward gauge-invariant method for the inclusion of the finite width is the factorization scheme introduced in Ref. [4], which consists in the multiplication of the matrix elements with a global resonance factor. However, for more complicated processes it becomes non-trivial to achieve a precision beyond LO. The fermion-loop scheme [5,6] exploits the fact that taking into account only closed fermion loops at the one-loop order allows to perform a gauge-invariant and gauge-independent resummation. By construction this method is restricted to leading-order predictions and to resonances that decay exclusively into fermions. The idea of a gauge-invariant resummation can be carried further by using the background-field method [7–9] which allows one to perform a Dyson summation without violating Ward identities [10]. While the resummed self-energies still depend on the quantum gauge parameter, this dependence can be fixed by definition, e.g. by using a specific gauge or the prescription of the pinch technique [11]. In practice these methods would require complete NNLO calculations to get NLO accuracy in the region of the resonance. The pole scheme proposed

[a] e-mail: denner@physik.uni-wuerzburg.de
[b] e-mail: jlang@physik.uni-wuerzburg.de

in Refs. [12,13] is based on the fact that both the location of the pole and the residue of the propagator of an unstable particle are gauge independent. It allows one to compute gauge-invariant matrix elements to arbitrary orders via a Laurent expansion around the complex pole. In practice this method gets quite involved in higher orders (see e.g. Ref. [14]), and usually only the leading terms in the Laurent expansion are taken into account, which is called the leading-pole approximation. Furthermore, effective-field theory can be used to describe unstable particles. In the method of Refs. [15,16] non-local gauge-invariant effective operators are introduced that allow for the gauge-invariant resummation of self-energies via appropriate choices of free parameters. In the effective-field-theory approach of Refs. [17,18] an expansion in the coupling constant and in the distance from the pole is performed simultaneously. This basically yields a field-theoretically elegant way to the pole approximation and can easily be combined with further expansions (see Ref. [19] for a recent application).

The most straightforward method to describe unstable particles in perturbation theory is the complex-mass scheme (CMS) [20–22]. It is fully gauge-invariant, valid everywhere in phase space, basically of the same complexity as a calculation for stable particles and applicable to higher orders in perturbation theory [23–26]. Finite widths are introduced by analytically continuing the renormalized mass parameters to appropriate complex values. The introduction of complex parameters immediately raises the question how unitarity is implemented in this scheme. Unitarity is not expected to be violated because the bare Lagrangian is left untouched and only the renormalization procedure is modified as compared to the standard treatment. Therefore any violation of unitarity should be beyond the order of perturbation theory taken into account completely. It has been shown by Veltman [27] within non-perturbative QFT that unitarity is fulfilled in a theory with unstable particles provided that the unstable particles are excluded from asymptotic states. Since the CMS provides a perturbative description of the full theory it should not violate unitarity, if observables are correctly computed in a valid perturbative regime. Moreover, the CMS guarantees exact gauge cancelations through gauge invariance order by order in perturbation theory. Unitarity within the CMS has been touched upon in Ref. [23]. Unitarity in the CMS in a model with a heavy vector boson interacting with a light fermion has been investigated at the one-loop level in Ref. [28].

The aim of this paper is to study unitarity in scalar field theories in the CMS. In Sect. 2 we briefly review unitarity and the largest-time equation in the case of stable particles and in Sect. 3 we summarize the CMS. In Sect. 4 we investigate the realization of unitarity in the CMS by constructing and exploiting a suitable largest-time equation for unstable particles.

2 Unitarity and Veltman's largest-time equation for stable particles

2.1 Unitarity

In the language of QFT unitarity means that the S matrix is unitary, i.e. $\mathcal{S}^\dagger\mathcal{S} = \mathbb{1}$. Separating the non-interacting contributions from $\mathcal{S}$ via $\mathcal{S} =: \mathbb{1} + \mathrm{i}\mathcal{T}$, one obtains the well-known relation

$$\mathcal{T}^\dagger\mathcal{T} = \mathrm{i}(\mathcal{T}^\dagger - \mathcal{T}) \tag{2.1}$$

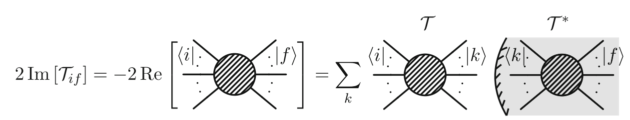
for the transition matrix $\mathcal{T}$. A simple consequence of unitarity is the optical theorem, which states that the imaginary part of a forward scattering amplitude $\mathcal{T}_{ii}$ is proportional to the total cross section:

$$\sigma_{\mathrm{tot}} = \text{flux factor} \times \mathrm{Im}\,[\mathcal{T}_{ii}]\,. \tag{2.2}$$

The connection between (2.1) and the optical theorem (2.2) is established when considering elements of the transition matrix with definite initial and final states,

$$\mathrm{i}(\mathcal{T}^*_{if} - \mathcal{T}_{fi}) = \sum_k \mathcal{T}^*_{kf}\mathcal{T}_{ki}, \tag{2.3a}$$

where the sum runs over all possible intermediate states k and total 4-momentum conservation is implied. In scalar theories, where $\mathcal{T}_{if} = \mathcal{T}_{fi}$, since the matrix elements do not depend on the direction of the momenta, or in general for forward scattering ($i = f$), the previous equation can be written as follows:



(2.3b)

The so-called shadowed region is given by $\mathcal{T}^*$ while the normal region is given by $\mathcal{T}$ and both transition amplitudes are connected by on-shell states which is visualized as a cut (dark hatched line). We call the equations (2.3) in the following the unitarity equation. Unitarity is verified by computing the left-hand side of the unitarity equation and comparing it to the right-hand side for all possible initial and final states. Direct computation of the left-hand side can be quite involved especially beyond the one-loop level, but with the help of cutting rules, which we introduce in the next section, the problem is solved theoretically and practically.

2.2 Veltman's largest-time equation and unitarity

The LTE can be seen as the analog to Cutkosky's cutting rules [29], but it is straightforward to derive and needs less mathematical tools. The derivation of the LTE for stable particles can be found, for instance, in Refs. [27,30], and it is based on a decomposition of the Feynman propagator in space-time representation. This decomposition is done, in the case of stable particles, in positive- and negative-time parts in such a way that positive (negative) time is connected to positive (negative) energy flow and vice versa.[1] Let $\Delta_{\mathrm{F}}(x-y)$ denote the Feynman propagator of a scalar particle in space-time representation

$$\Delta_{\mathrm{F}}(x-y)=\frac{1}{(2\pi)^4}\int \mathrm{d}^4 p\,\frac{\mathrm{e}^{-\mathrm{i}p(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}, \tag{2.4}$$

where $\mathrm{i}\epsilon$ $(\epsilon>0)$ is an infinitesimal imaginary part that ensures causality, then the decomposition is the following:

Decomposition theorem: *There exist functions $\Delta^{\pm}$ with the properties*

$$\begin{aligned}\Delta_{\mathrm{F}}(x_i-x_j)&=\theta(x_i^0-x_j^0)\Delta^{+}(x_i-x_j)\\&\quad+\theta(x_j^0-x_i^0)\Delta^{-}(x_i-x_j), \qquad (2.5)\\ \Delta^{\pm}(x_i-x_j)&=-(\Delta^{\mp}(x_i-x_j))^{*}=\Delta^{\mp}(x_j-x_i),\end{aligned}$$

where $\Delta^{+}(x_i-x_j)$ and $\Delta^{-}(x_i-x_j)$ correspond to positive and negative energy flow, respectively.

In Fourier space they take the simple form

$$\Delta^{\pm}(p,m^2)=\mp 2\mathrm{i}\pi\theta(\pm p_0)\delta(p^2-m^2). \tag{2.6}$$

Given such a decomposition, one can define extended Feynman rules:

The underline operation: *Given a Feynman diagram $\mathcal{F}$ defined by a set of vertices $\{x_i\}$ and corresponding couplings $\{g_i\}$, we define new diagrams where one or more of the space-time points x_i can be underlined*, i.e. $x_i\to\underline{x_i}$. *This operation shall have the following consequences for propagators connecting the vertices in the original diagram:*

- $\mathrm{i}\Delta_{ki}=\mathrm{i}\Delta_{\mathrm{F}}(x_k-x_i)$ *is unchanged if x_k, x_i are unchanged,*
- $\mathrm{i}\Delta_{ki}$ *is transformed as* $\mathrm{i}\Delta_{ki}\to\mathrm{i}\Delta_{ki}^{+}=\mathrm{i}\Delta^{+}(x_k-x_i)$ if $x_k\to\underline{x_k}$, *but x_i remains unchanged,*
- $\mathrm{i}\Delta_{ki}$ *is transformed as* $\mathrm{i}\Delta_{ki}\to\mathrm{i}\Delta_{ki}^{-}$ *if $x_i\to\underline{x_i}$, but x_k remains unchanged,*
- $\mathrm{i}\Delta_{ki}$ *is transformed as* $\mathrm{i}\Delta_{ki}\to-\mathrm{i}\Delta_{ki}^{*}$ *if two connected space-time points x_k, x_i are underlined,*
- *any underlined space-time point implies a factor -1 for the corresponding vertex*, i.e. *if $x_k\to\underline{x_k}$, then the corresponding coupling is replaced as* $\mathrm{i}g_k\to-\mathrm{i}g_k$.

At the level of Feynman diagrams the underline operation is indicated by a circle ◯ at the corresponding underlined space-time points. The rules stay the same for couplings with imaginary part; in particular, we stress that the coupling g_i is not complex-conjugated for underlined x_i.

As has been shown by Veltman in Ref. [27] (see also Ref. [30]), the following equation can be derived from these rules:[2]

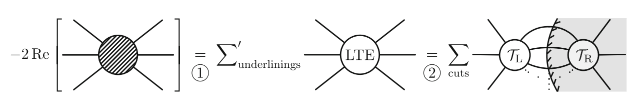
Largest-time equation: *Given a Feynman diagram $\mathcal{F}$ defined by a set of vertices $\{x_i\}$ and corresponding couplings $\{g_i\}$, if the Lagrangian is hermitian and all propagators fulfill the decomposition theorem, then the following equation holds:*

$$\sum_{\text{underlinings}}\mathcal{F}(x_1,\ldots\underline{x_i},\ldots,\underline{x_j},\ldots,x_N)=0, \tag{2.7}$$

where the sum runs over all possibilities of underlining elements x_i (called LTE diagrams in the following). In total there are 2^N contributions where N is the number of vertices.

We note that the LTE holds both for truncated or non-truncated diagrams, and as the prescription (2.7) is linear it does also hold for sums of diagrams and for complete amplitudes $\mathcal{T}$.

The unitarity equation (2.3) is recovered by extracting two contributions from the LTE, namely the one where none of the vertices are underlined and the one where all of them are underlined, i.e. $\mathcal{T}(x_1,\ldots,x_i,\ \ldots,x_N)$ and $\mathcal{T}(\underline{x_1},\ldots,\underline{x_i},\ldots,\underline{x_N})$. These two contributions match $\mathrm{i}\mathcal{T}_{fi}$ and $-\mathrm{i}\mathcal{T}_{if}^{*}$ and shifting them to the other side of equation (2.7) we obtain the identity visualized in step ① of (2.8), where the primed sum over underlinings stands for the sum over all possible LTE amplitudes except for $\mathrm{i}\mathcal{T}_{fi}$ and $-\mathrm{i}\mathcal{T}_{fi}^{*}$



(2.8)

The right-hand side of (2.3), i.e. step ② of (2.8), is obtained by applying the kinematic constraints, i.e. the θ and δ functions, imposed by the explicit solutions $\Delta^{\pm}$ (2.6). LTE amplitudes not satisfying these constraints vanish and the remaining ones can be written in terms of all possible ways of connecting amplitudes $\mathcal{T}_{\mathrm{L}}$ with complex conjugated amplitudes $\mathcal{T}_{\mathrm{R}}^{*}$ via cut propagators (step ②). The cut propaga-

[1] The energy-flow direction is related to the sign of p^0, where p^0 is the zeroth component of the four-momentum. We prefer to speak about positive- and negative-time parts instead of positive- and negative-energy parts because of the generalization to unstable particles introduced in Sect. 4.2.

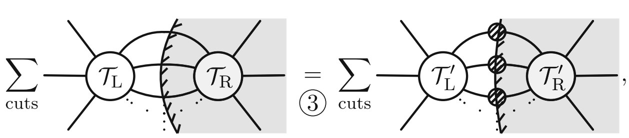
[2] Equation (2.7) is actually a consequence of Veltman's LTE. For the sake of simplicity we use the term LTE for this equation in the following.

tors $\Delta^{\pm}$ are the solutions (2.6) which represent propagators where one space-time point is not underlined while the other is. Thus, each non-vanishing LTE amplitude has two well-defined regions, a region with the usual Feynman rules which is always connected to the incoming particles and a region with the "complex-conjugated" Feynman rules (underlined vertices) which is always connected to outgoing particles. Put in other words: four-momentum conservation and the given values of external four-momenta forbid certain contributions to the LTE, and the contributions left are the ones where the energy flows from incoming particles to outgoing particles as it is required by the unitarity equation (2.3). The property of LTE amplitudes to fall apart into two separate regions, thus justifying the representation of ②, is called *cut structure* in the following and reviewed in Sect. 4.3.1.

Cutkosky's cutting rules: The underline operation together with the LTE is equivalent to Cutkosky's cutting rules, namely that the discontinuity of an amplitude is obtained by replacing propagators in all possible ways by on-shell propagators (2.6), but it is constrained in such a way that the energy flows from the initial to the final states. For more details, in particular for the derivation of Cutkosky's cutting rules, we refer to the original reference [29]. In the following the terminologies Cutkosky's rules, cutting rules, and LTE with the usual on-shell cut propagator (2.6) are used as synonyms.

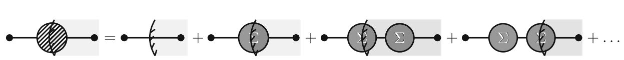
2.3 A decomposition for dressed propagators

The applicability of the LTE is not restricted to amplitudes and can be applied to Green's functions. A cut amplitude can be expressed in terms of cut two-point functions which requires a decomposition, similar to the case of the usual Feynman propagator, for two-point functions. The decomposition can be achieved for dressed propagators via the Källén–Lehmann representation as has been shown by Veltman in Ref. [27]. Applying this idea to the unitarity equation leads to a reinterpretation of the right-hand side of (2.8) where simple cut propagators are replaced by cut two-point functions



(2.9)

where $\mathcal{T}_{\rm L,R}$ denote the subamplitudes on the left- and right-hand sides of the cut. The cut two-point function is given by



(2.10)

and the dotted lines in (2.9) indicate that we can have an arbitrary number of cut propagators. The equality ③ holds only for the sum of all cut amplitudes. From the perturbative point of view the equality follows by inserting the cut two-point function on the right-hand side of (2.9) and by identifying $\mathcal{T}_{\rm L,R}$ with $\mathcal{T}'_{\rm L,R}$ supplemented by all non-cut parts of (2.10).

3 The CMS

When dealing with gauge theories it is crucial to guarantee gauge invariance which is more involved when unstable particles are present. As pointed out in the introduction, various methods have been developed to describe unstable particles in perturbation theory, but most are only valid near the resonance and lack validity in general phase-space regions. In contrast, the CMS is valid in the full phase space. Its underlying idea is an analytic continuation in the masses of the unstable particles. Being analytic relations not involving complex conjugation, the Ward identities are not violated by such a modification. In practice, the renormalized Lagrangian is rewritten by replacing any appearing mass corresponding to an unstable particle with the complex one in such a way that the bare Lagrangian is not changed. In a way the CMS is just a renormalization scheme with complex renormalization constants.

We sketch the procedure: In the first step renormalized parameters are introduced. Let m_0 denote the bare mass of an unstable particle, then introduce

$$m_0^2 =: \mu^2 + \delta\mu^2. \tag{3.1}$$

The complex mass μ^2 is attributed to the propagator and resummed, while the counter term $\delta\mu^2$ is treated as a vertex and not resummed.

Thus, the LO propagator in the CMS reads

$$\Delta_{\rm F}(x-y,\mu) := \frac{1}{(2\pi)^4}\int {\rm d}^4 p\, \frac{{\rm e}^{-{\rm i}p(x-y)}}{p^2-\mu^2}, \tag{3.2}$$

or in momentum space

$$\Delta_{\rm F}(p,\mu) = \frac{1}{p^2-\mu^2}. \tag{3.3}$$

The usual causality iϵ prescription [see (2.4)] becomes irrelevant owing to the finite imaginary part of $\mu^2 = M^2 - {\rm i}\Gamma M$.

The procedure implies that the mass counter terms are complex. Since the bare mass is real, the following consistency equation holds:

$$\mathrm{Im}[\mu^2] = -\mathrm{Im}[\delta\mu^2]. \tag{3.4}$$

Couplings that are purely real in the conventional framework become complex in the CMS if they are related to the masses, which is, for instance, the case for the electroweak mixing angle in the Glashow–Salam–Weinberg theory [20,22].

We have to employ suited renormalization conditions in order to fix the finite part of the parameters. Usually this is done in the on-shell scheme which is distinguished by the fact that the renormalized parameters are equal to physical observables. More concretely, one requires that the renormalized two-point function of a stable particle near its mass $p^2 = m^2$ is given by the Feynman propagator (2.4). This condition does fix both the mass renormalization and the field renormalization (see e.g. Ref. [31]). The on-shell scheme can be extended to the case of unstable particles, and the appropriate renormalization conditions read [20,22]:

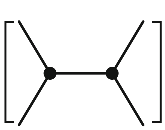
$$\Sigma_{\mathrm{R}}(p^2)|_{p^2=\mu^2} = 0, \quad \Sigma'_{\mathrm{R}}(p^2)|_{p^2=\mu^2} = 0. \tag{3.5}$$

Here Σ_{R} denotes the renormalized self-energy of the unstable particle and Σ'_{R} is the corresponding renormalized self-energy differentiated with respect to p^2. The renormalization conditions (3.5) together with the requirement that the bare Lagrangian is real, yielding consistency equations like (3.4), outline a gauge-invariant renormalization procedure. Apart from the validity of Ward identities one must make sure that the renormalization conditions do not introduce a gauge dependence. Given the fact that the complex pole is gauge independent, the renormalization point and the renormalization condition (3.5) are gauge independent. Those proofs were carried out by Stuart [1], Sirlin [2,3,32], Gambino and Grassi [33] and Grassi et al. [34].

Even though the renormalization conditions are similar to the ones in the on-shell scheme the difference may be significant as it is the case in the SM for the mass prediction of the W and Z bosons [35]. In view of gauge theories and physical observables, the complex pole is more than a theoretical construct and should be seen as the analog to the mass for stable particles. For a discussion we refer to Ref. [36].

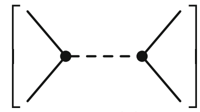
4 Unitarity in the CMS for scalar field theories

In the CMS the Cutkosky rules are not valid in the sense that their application does not yield the same result as one would get by direct computation of the left-hand side of the unitarity equation (2.3). For instance, the Cutkosky rules require that the discontinuity of the tree-level s-channel diagram vanishes for $s \neq m^2$,

$-2\,\mathrm{Re}$ 
stable

$$= \begin{cases} 0 & \text{if} \quad s \neq m^2 \\ \text{undefined} & \text{if} \quad s = m^2 \end{cases} . \tag{4.1}$$

Replacing the stable particle with an unstable one, the direct computation yields

$-2\,\mathrm{Re}$ 
unstable

$$= \frac{\Gamma M}{(s-M^2)^2 + (\Gamma M)^2} \neq 0 \quad \forall s. \tag{4.2}$$

In view of the LTE the reason is that the preconditions are not fulfilled and we cannot use the cutting rules for a propagator without having shown that there is a valid decomposition (2.5).

As a consequence of the analytical continuation of the S matrix to complex masses algebraic relations are untouched, but operations where complex conjugation is involved are no longer preserved as is the case for the unitarity equation. The CMS guarantees gauge invariance, but it is no longer clear how unitarity is implemented. Veltman has shown [27] that for a super-renormalizable theory the S matrix in non-perturbative QFT is unitary on the Hilbert space spanned by only stable particles,

$$\mathrm{i}(\mathcal{T}^*_{if} - \mathcal{T}_{fi}) = \sum_{|k\rangle \,\in\, \text{stable particles}} \mathcal{T}^*_{kf}\mathcal{T}_{ki}. \tag{4.3}$$

Starting from the Källén–Lehmann representation for unstable particles which, in contrast to stable particles, lacks a one-particle pole on the real axis, he showed unitarity by deriving a LTE for dressed propagators.

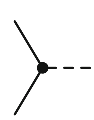
We apply this idea to the CMS in perturbative QFT, derive a corresponding LTE and show that amplitudes $\mathcal{T}$ computed within the CMS at a perturbative order g^n for a fixed kinematic configuration are unitary up to higher orders,

$$\mathrm{i}(\mathcal{T}^*_{if} - \mathcal{T}_{fi}) = \sum_{|k\rangle \,\in\, \text{stable particles}} \mathcal{T}^*_{kf}\mathcal{T}_{ki} + \mathcal{O}(g^{n+1}), \tag{4.4}$$

where two sources of higher orders emerge. The first source are kinematically suppressed terms in the LTE, which are effectively suppressed by $\mathcal{O}(\Gamma) \sim \mathcal{O}(g^2)$, and the other one are corrections due to the resummation of finite-width terms in the CMS. Both turn out to result in non-relevant perturbative corrections of order $\mathcal{O}(g^{n+1})$.

4.1 Sketch of the proof

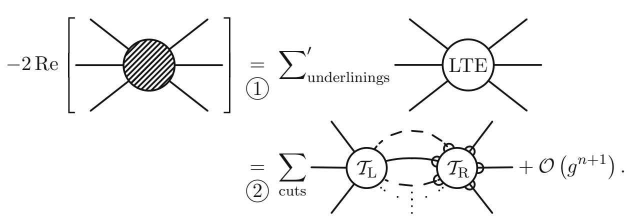
We restrict the discussion to a simple scalar toy model. It consists of an unstable real scalar field ϕ with mass squared $\mu^2 = M^2 - \mathrm{i}\Gamma M$ and a stable real scalar field χ with mass m and the interaction

$\mathcal{L}_{\mathrm{I}} = \frac{g}{2!}\phi\chi^2 \quad \Leftrightarrow$  $= \mathrm{i}g\,.$ (4.5)

Owing to the required resummation of the width in the resonant propagators there is no unique perturbative order for Feynman diagrams or matrix elements in the CMS. The CMS propagator $1/(s-\mu^2)$ affects the perturbative order (it is of order $1/M\Gamma$ in the resonance region but of order one otherwise), but has no Taylor expansion near the resonance, i.e. for $s \approx M^2$. Since ΓM is of order $g^2 M^2$, the change in perturbative order occurs near the resonance. The lack of a unique perturbative order of amplitudes complicates the investigation of perturbative unitarity in the CMS. However, we can speak of relative orders in the sense that if the left-hand side of the unitarity equation has a phase-space-dependent order then also the right-hand side does, and the difference in the orders is independent of the phase-space region.

Our strategy concerning perturbative expansion is as follows: We rely on perturbation theory in the CMS. We do never expand resonant propagators in the amplitudes. We do, however, expand resonant cut propagators $\Delta^{\pm}$ (if not stated otherwise, Δ denotes the propagator of the unstable particle in the following). In our model, the width of the unstable particle is of order of the coupling squared, i.e. $\mathcal{O}(\Gamma) = \mathcal{O}(g^2)$. This allows one to define a consistent power counting in the region of a resonance and away from it. While the absolute power counting depends on the phase-space region, the relative power counting does not.

We here briefly sketch our derivation of Veltman's unitarity equation (4.3) in the CMS, to be elaborated in the following subsections. It is done in four steps, similarly to the three steps ①, ② and ③ of Eqs. (2.8) and (2.9) described in Sect. 2.3. In Sect. 4.2 we construct a decomposition of the CMS propagator of the form (2.5). Then it follows immediately that we can compute the left-hand side of (4.3) via the LTE [step ① in (4.6)] as long as we consider interactions with couplings that are either real or have a corresponding complex-conjugated counterpart in the interaction part of the Lagrangian,[3]



(4.6)

Otherwise the LTE is still valid, but the amplitude with all space-time points underlined is not any more equal to the complex-conjugated amplitude, and the left-hand side of the unitarity equation can no longer be related to contributions to the LTE.

The preconditions for this identification are automatically fulfilled for interaction vertices if the bare Lagrangian is real. The argument fails for the imaginary mass counter term $\mathrm{Im}[\delta\mu^2]$ because the corresponding counter part resides in the resummed propagator, thus, we have a mismatch in the perturbative order and we have to take care of this contribution differently.

In Sect. 4.3.1 we review kinematic arguments needed for identifying cut contributions and we investigate the cut structure and changes when unstable particles are present.

In Sect. 4.3.2 we explain how to compute the LTE amplitudes when taking into account the imaginary mass counter term. We show that the LTE amplitudes are obtained in the usual way except for the cut contributions where each mass counter term insertion must be rewritten appropriately.

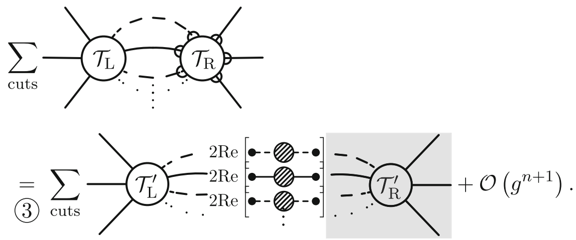
In Sect. 4.3.3 we show to all orders, using the representation introduced in Sect. 4.3.2, that LTE amplitudes within the CMS split into a normal region $\mathcal{T}$ and a complex-conjugated region $\mathcal{T}^*$, up to terms of higher perturbative order, i.e. we show that step ② in (4.6) holds. The circles attached to $\mathcal{T}_{\mathrm{R}}$ in (4.6) indicate that $\mathcal{T}_{\mathrm{R}}$ is completely underlined (in the sense of Sect. 2.2). The plain (dashed) lines represent stable (unstable) particles, and the dots indicate that we can have an arbitrary number of cut propagators. In order to identify higher orders we need to expand in Γ/M. We do not perform the expansion for the whole amplitude, but only for the cut propagators of the unstable particles $\Delta^{\pm}$. We proof that LTE amplitudes containing kinematically suppressed, or equivalently, non-resonant $\Delta^{\pm}$ do not contribute to the unitarity equation in the considered order but are always of higher, irrelevant order.

In Sect. 4.3.4 we investigate resonant cut contributions. Resonances in $\Delta^{\pm}$ play an essential role as they represent kinematically allowed channels found on the right-hand side of the unitarity equation. We point out that it is important to distinguish the resonances appearing in the matrix elements $\mathcal{T}_{\mathrm{L}}$, $\mathcal{T}_{\mathrm{R}}$ and resonant cut contributions ($\Delta^{\pm}$) in view of unitarity. The former do always appear in the same way on both sides of the unitarity equation, thus enhancing both sides of the unitarity equation equally and we do not need to consider them at all. We stress that we only expand unstable cut propagators, but never uncut propagators. Expanding unstable cut propagators at leading order in Γ/M, we find that a resonant $\Delta^{\pm}$ of an unstable particle is just a δ function [see (4.12)] which seems to be in conflict with the fact that unstable particles should not appear as asymptotic states. However, we show that the δ function can be interpreted as a

[3] For real couplings that become complex in the CMS (e.g. the mixing angles in the standard model), the LTE is still valid because their complex-conjugated part is located in the corresponding counter term which follows from the fact that the bare coupling is real. Thus, any unitarity-violating terms from those complexified couplings are trivially of higher order.

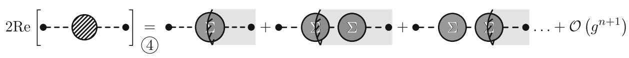
cut self-energy of the unstable particle, thus being consistent with Veltman's statement.

In order to proof unitarity in the CMS beyond one loop we reformulate the LTE in terms of nested LTE amplitudes of two-point functions in Sect. 4.3.5 yielding the representation on the right-hand side of the following equation:



(4.7)

The leading-order result of our expansion (see Sect. 4.3.4) serves as the induction start for showing unitarity in general. From (4.7) unitarity is implied given that



(4.8)

holds, and we recover the same expression as for stable particles [(2.9) of Sect. 2.3]. In the final step we make use of the fact that the CMS partially resums contributions and this partial resummation appears in the mass counter term. Rearranging the LTE in terms of two-point functions, unitarity-violating terms cancel between self-energy terms and mass counter terms (see Sect. 4.3.5).

4.2 Decomposition for the CMS propagator

We start with the construction of a propagator decomposition in the case of the CMS. Since the decomposition is not unique, we list our assumptions which led to it. Basically, one decomposes the propagator into positive- and negative-time parts and adds something to these parts which is zero for positive and negative times, respectively. Working out this idea, one realizes that the allowed transformations have always, in Fourier space, the form of an advanced and retarded propagator $\Delta_{\mathrm{A/R}}$. Hence, our approach consists of defining meromorphic functions $\Delta_{\mathrm{A/R}}$ with similar pole structure as in the case of stable particles, and among possible restrictions Occam's Razor suggests

$$\begin{aligned}
&\Delta_{\mathrm{F}}(p,\mu) - \Delta_{\mathrm{A}}(p^0,\mathbf{p},M,\Gamma) \stackrel{!}{=} \Delta^{+}(p^0,\mathbf{p},M,\Gamma),\\
&\Delta_{\mathrm{F}}(p,\mu) - \Delta_{\mathrm{R}}(p^0,\mathbf{p},M,\Gamma) \stackrel{!}{=} \Delta^{-}(p^0,\mathbf{p},M,\Gamma),\\
&\int \mathrm{d}p^0\ \Delta_{\mathrm{A/R}}(p^0,\mathbf{p},M,\Gamma)\mathrm{e}^{\pm \mathrm{i}p^0|x^0|} = 0.
\end{aligned} \tag{4.9}$$

The function $\Delta_{\mathrm{A/R}}$ must be chosen such that $\Delta^{\pm}$ fulfills the decomposition theorem (2.5). The third equation is the condition that the advanced/retarded propagator has only poles in the upper/lower complex plane as it should be. Consequently, we have the same situation as in the case of stable particles, namely $\theta(\pm x^0)\mathrm{FT}[\Delta_{\mathrm{A/R}}](x) = 0$, where FT denotes the Fourier transformation. Furthermore, we require that, similar to the case of stable particles, the retarded propagator turns into the advanced propagator by complex conjugation and vice versa, which is our last assumption

$$\Delta_{\mathrm{A}}(p^0,\mathbf{p},M,\Gamma) = (\Delta_{\mathrm{R}}(p^0,\mathbf{p},M,\Gamma))^{*}. \tag{4.10}$$

Given these restrictions one can easily derive the unique solutions for $\Delta^{\pm}$. In Fourier space they read

$$\Delta^{\pm}(p,\mu) = \mathrm{i}\,\mathrm{Im}\left[\frac{1}{\hat{p}^0(p^0 \mp \hat{p}^0)}\right] \quad \text{with} \quad \hat{p}^0 = \sqrt{\mathbf{p}^2+\mu^2}. \tag{4.11}$$

As a first but very important result one verifies that in the limit $\Gamma \to 0^+$ our solutions turn into the stable ones, i.e.

$$\lim_{\Gamma\to 0^+} \Delta^{\pm}(p^2,\mu^2) = \mp 2\pi\mathrm{i}\theta(\pm p^0)\delta(p^2-M^2). \tag{4.12}$$

In view of consistency, this means that there is a smooth transition from unstable to stable propagator as the mass M^2 tends below the kinematic limit of instability. On the other hand, this result tells us that in a perturbative expansion in Γ the leading-order "cut" contribution is equal to the cut contribution of a stable particle with the same mass. We note that $\Delta^{\pm}(p,\mu)$ for finite Γ does neither involve a δ nor a θ function. Thus, energy can flow in both directions and the realization of causality is more involved for unstable particles.

Apparently, two problems appear:

- Given an S matrix, an expansion in small Γ can often be performed only in a distributional sense, even though perturbation theory predicts $\mathcal{O}(\Gamma) = g^2$, where g is the coupling constant. For instance, for the s-channel production of an unstable particle the width is crucial for the finiteness of the result. The question is when we are allowed to do a naive expansion or when is it actually necessary since we only want to verify the unitarity equation (2.3).
- At first sight the fact that the cutting rules for the CMS propagator for $\Gamma \to 0^+$ coincide with the ones for stable particles might interfere with Veltman's result, namely that only stable particles appear as asymptotic states in the unitarity equation (4.3).

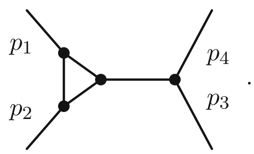
The two points do not pose any problems, as we show in the upcoming sections.

4.3 Cutting rules for unstable particles

4.3.1 Kinematic restrictions

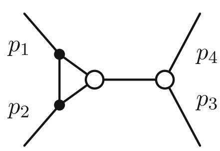
The cutting rules are a special case of the LTE relations where many terms in (2.7) do not contribute because the S matrix underlies physical constraints such as positive energy flow and real masses. These constraints reappear in the LTE amplitudes in form of δ and θ functions.

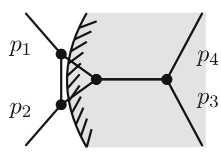
The situation is similar for stable and unstable particles, and we consider stable particles first. In our convention the incoming particles are on the left and the outgoing ones on the right. As an example consider the following diagram in a scalar ϕ^3 theory:

$\mathcal{F}\left(p_1, p_2, p_3, p_4\right) =$ 

. (4.13)

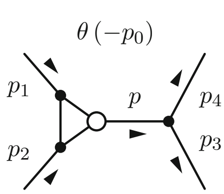
If we sum over all possible underlinings the result equals zero except for one contribution which is immediately recognized as a cut

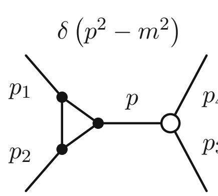
$-2\,\mathrm{Re}\left[\mathcal{F}\right] =$ 

$:=$ 

. (4.14)

The example shows that the non-vanishing contributions of the LTE for stable particles split into two separate regions where the normal part ($\mathcal{T}$) is given by the black dots, while the complex-conjugated part ($\mathcal{T}^*$) is given by the white circles. In the following we call this property the *cut structure*. This means, in particular, that for stable particles the LTE and simple kinematic arguments lead to the unitarity equation (2.3). Examples of vanishing LTE terms are



$= 0\,,$ 

$= 0\,.$ (4.15)

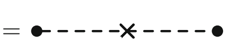
In the first term of (4.15) the cut structure is violated, i.e. there are no two well-defined regions, and as a consequence at some vertex in the amplitude the required energy-flow direction is opposed to the physical energy flow which is from the left to the right, and no energy can be transferred to the final state. The second amplitude vanishes because the cutting rules require the intermediate particle to be on-shell which is impossible for stable particles since $m^2 < p^2 = (p_1 + p_2)^2$.

When CMS propagators are involved one would like to have, in particular, the cut structure of non-vanishing LTE terms, but this is a priori not given. For instance, the first term in (4.15) does not vanish when the cut propagator is replaced by the corresponding CMS propagator. These cut-structure-violating contributions come from the fact that for the CMS propagator $\Delta^{\pm}(p, \mu)$ (4.11) there is neither a $\theta(\pm p^0)$ nor a $\delta(p^2 - M^2)$ but smoothed functions instead. The smoothing does no longer enforce the same strict kinematic constraints as for stable particles. Nevertheless, these contributions are suppressed by at least a factor $\Gamma/M \sim g^2$, and one obtains the same behavior for unstable particles in a perturbative sense, meaning that those LTE terms violating the cut structure are always of higher order in the coupling constant. While the first LTE diagram in (4.15) is also perturbatively suppressed for $p^2 \sim M^2$, the second one does not violate the cut structure and, in fact, in the case of unstable particles its contribution is relevant. Such contributions are discussed in Sects. 4.3.4 and 4.3.5.

Yet, the argument is incomplete since suppressed terms can become relevant as one takes into account higher perturbative orders, i.e. they can be of the same order as higher quantum corrections. The next chapter is devoted to include the imaginary mass counter term in the LTE. We discuss how to simplify LTE relations and we show that the imaginary mass counter term is responsible for the fact that contributions being negligible at a certain perturbative order, stay negligible even if the calculation is extended to higher orders (see Sect. 4.3.3 for non-resonant and Sects. 4.3.4 and 4.3.5 for resonant propagators).

4.3.2 Including the imaginary mass counter term

A proper description of unstable particles requires the resummation of self-energy contributions resulting in a non-zero imaginary part in the LO propagator. On the other hand, gauge invariance requires that the imaginary counter part of the complex mass enters the Feynman rules. It is not possible [37] to include such a coupling in the LTE relations. However, this is not necessary as we discuss in the following. Consider the insertion of a $\mathrm{i}(-\mathrm{i}\Gamma M)$ coupling between two CMS propagators in momentum space,

$(\mathrm{i}\Delta)\left(-\mathrm{i}^2\Gamma M\right)(\mathrm{i}\Delta) =$  . (4.16)

This insertion can always be reduced to the usual propagator via simple differentiation with respect to ΓM

$$(\mathrm{i}\Delta)(-\mathrm{i}^2\Gamma M)(\mathrm{i}\Delta) = -\Gamma M \frac{\partial}{\partial \Gamma M}\mathrm{i}\Delta, \tag{4.17}$$

and, as becomes clear below, it is important that ΓM is real which is true by construction.

For an arbitrary amplitude $\mathcal{T}$, the left-hand side of the unitarity equation (2.3) is computed as follows: Every insertion of $\mathrm{i}(-\mathrm{i}\Gamma M)$ can be generated by differentiating specific CMS propagators according to (4.17). Consequently, any amplitude with $(-\mathrm{i}\Gamma M)$ insertion in the CMS can be generated from the diagrams in an amplitude where $\mathrm{i}(-\mathrm{i}\Gamma M)$ is missing. Consider the diagrams $\mathcal{F}^\tau$ obtained from $\mathcal{T}$ by setting the imaginary mass counter term ΓM to zero, but keeping the resummed counter part in the propagator,

$$\mathcal{T}|_{\Gamma M=0} = \sum_\tau \mathcal{F}^\tau. \tag{4.18}$$

We denote the set of propagators which are linked to a mass counter term by $\Omega := \{\Omega_{\tau,i}\}$ where i identifies the propagator and τ the diagram in $\mathcal{T}$ which contains the propagator i. Multiple insertions of $\mathrm{i}(-\mathrm{i}\Gamma M)$ are collected and the number of successive insertions is denoted by n^i_τ. We transform each propagator i in τ via $\Phi_\Omega : \Gamma M \to \Gamma M + \omega^i_\tau$ in its denominator, and the amplitude $\mathcal{T}$ is then generated via

$$\mathcal{T} = \sum_{\tau\in\Omega}\prod_{i\in\tau}\frac{1}{n^i_\tau!}\left(-\Gamma M\frac{\partial}{\partial\omega^i_\tau}\right)^{n^i_\tau}\mathcal{F}^\tau_{\Phi_\Omega}\Bigg|_{\omega^i_\tau=0}, \tag{4.19}$$

where $\mathcal{F}^\tau_{\Phi_\Omega}$ is the diagram $\mathcal{F}^\tau$ with the propagators transformed according to Φ_Ω.

By construction $\mathcal{F}^\tau_{\Phi_\Omega}$ is free of imaginary mass counter terms and in the case of a scalar theory, as we consider it here, we can simply commute differentiation with taking the real part. Thus, we can directly apply the LTE on $\mathcal{F}^\tau_{\Phi_\Omega}$ and after performing the (real) differentiation we obtain the real part of $\mathcal{T}$ via (4.19). This representation implies that the cutting rules in $\mathcal{T}, \mathcal{T}^*$ stay the same even if the imaginary mass counter term is included. To see this consider the case when we have a CMS propagator and at least one insertion of $\mathrm{i}(-\mathrm{i}\Gamma M)$ in an amplitude $\mathcal{T}$

$$\mathcal{T} = \int \tilde{\mathcal{T}}\ \mathrm{i}g\ \mathrm{i}\Delta(p^2, M^2 - \mathrm{i}\Gamma M)\ \mathrm{i}(-\mathrm{i}\Gamma M) \times \mathrm{i}\Delta(p^2, M^2 - \mathrm{i}\Gamma M)\ \mathrm{i}g, \tag{4.20}$$

where $\tilde{\mathcal{T}}$ denotes the amplitude $\mathcal{T}$ with the two-point function with the imaginary mass counter term insertion omitted and the two couplings $(\mathrm{i}g)^2$ connecting $\tilde{\mathcal{T}}$ and the two-point function between those couplings removed. The $\int$ indicates that the propagator's momentum p^2 may be integrated. Rewriting the insertion one is left with

$$\mathcal{T} = -\Gamma M\frac{\partial}{\partial\omega}\underbrace{\int \tilde{\mathcal{T}}\ \mathrm{i}g\ \mathrm{i}\Delta(p^2, M^2 - \mathrm{i}(\Gamma M + \omega))\ \mathrm{i}g}_{=\mathcal{T}^\tau_{\Phi_\Omega}}\Big|_{\omega=0}. \tag{4.21}$$

Applying the LTE, the propagator $\mathrm{i}\Delta$ either stays the same ($\in \mathcal{T}$), transforms into $-\mathrm{i}\Delta^*$ ($\in \mathcal{T}^*$), or belongs to the cutting region

$$\begin{aligned}0 = &-\Gamma M\frac{\partial}{\partial\omega}\int\left(\sum_{\text{underlinings}}\tilde{\mathcal{T}}\right)\\ &\times(\mathrm{i}g)^2[\mathrm{i}\Delta(p^2, M^2 - \mathrm{i}(\Gamma M + \omega))\\ &+(-1)^2\times(-\mathrm{i})\times\Delta^*(p^2, M^2 - \mathrm{i}(\Gamma M + \omega))\\ &+(-1)\times\mathrm{i}\Delta^+(p^2, M^2 - \mathrm{i}(\Gamma M + \omega))\\ &+(-1)\times\mathrm{i}\Delta^-(p^2, M^2 - \mathrm{i}(\Gamma M + \omega))]|_{\omega=0}.\end{aligned} \tag{4.22}$$

While the first term in (4.22) corresponds to the propagator left of the cut and the second term to the one right of the cut, the third and fourth terms represent cut propagators with dominantly positive or negative energy flow. The sum over the underlinings of $\tilde{\mathcal{T}}$ represents the LTE equation of the truncated amplitude[4] $\tilde{\mathcal{T}}$. If the propagator is in the normal region we recover the result (4.17) after applying the differentiation

$$\begin{aligned}&-\Gamma M\frac{\partial}{\partial\omega}\mathrm{i}\Delta(p^2, M^2 - \mathrm{i}(\Gamma M + \omega))\Big|_{\omega=0}\\ &= \mathrm{i}\Delta(p^2, \mu)\mathrm{i}(-\mathrm{i}\Gamma M)\mathrm{i}\Delta(p^2, \mu).\end{aligned} \tag{4.23}$$

For the propagator in the complex-conjugated region $-\mathrm{i}\Delta^*(p, \mu)$ we can work out the signs leading to

$$\begin{aligned}&-\Gamma M\frac{\partial}{\partial\omega}(-\mathrm{i}\Delta^*(p^2, M^2 - \mathrm{i}(\Gamma M + \omega)))\Big|_{\omega=0}\\ &= (-\mathrm{i}\Delta^*(p^2, \mu))\mathrm{i}(-\mathrm{i}\Gamma M)(-\mathrm{i}\Delta^*(p^2, \mu)).\end{aligned} \tag{4.24}$$

Consequently, we obtain cutting rules in the regions $\mathcal{T}, \mathcal{T}^*$ for $\mathrm{i}(-\mathrm{i}\Gamma M)$ which coincide with the usual Feynman rules, namely that $\mathrm{i}(-\mathrm{i}\Gamma M)$ is treated purely real.

Remember that we cannot deal with *arbitrary* complex couplings in the LTE unless we give up the relation between LTE and unitarity. Imaginary couplings do not turn into their complex conjugates according to the last underlining rule (see Sect. 2.2) which is necessary for identifying the left-hand side of the unitarity equation, but we have shown that $\mathrm{i}(-\mathrm{i}\Gamma M)$ transforms correctly by other means. The result is

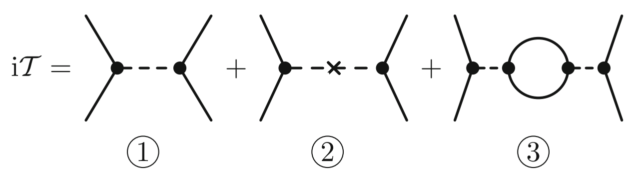
[4] The amplitude $\tilde{\mathcal{T}}$ is missing two vertices compared to $\mathcal{T}$, not counting the imaginary mass counter term. Therefore, the total number of LTE amplitudes corresponding to $\tilde{\mathcal{T}}$ is 2^{n-2} where n is the number of vertices in $\mathcal{T}$. Multiplying with the four underlinings of the propagator we recover the 2^n LTE amplitudes of $\mathcal{T}$.

true for more than just one insertion which can be shown by working out the signs for multiple differentiation. The case that the propagator of the unstable particle is on the cut is discussed in the following sections.

4.3.3 Non-resonant contributions of unstable particle propagators

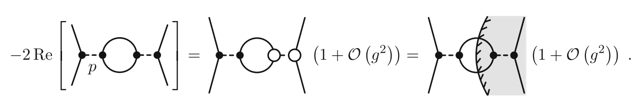
We come back to the question whether contributions, which in the case of stable particles vanish because of kinematic constraints, can actually contribute in the case of unstable particles. In this subsection we show that the $\mathrm{i}(-\mathrm{i}\Gamma M)$ insertions make sure that contributions that vanish for stable particles never become relevant for unstable ones (which are not resonant).

Consider the amplitude



(4.25)

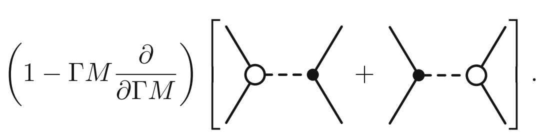
and assume $\left|p^2 - M^2\right| \gg M\Gamma$ (off resonance), then the order of accuracy of the amplitude is $\mathcal{O}(\mathcal{M}) = g^4$. Computing the unitarity equation (2.3), the leading contribution to the left-hand side results from ③



(4.26)

The higher-order contributions $\mathcal{O}(g^2)$ are of the type of (4.15), i.e. they either violate the cut structure or are further suppressed [owing to $p^2 \neq M^2$ and (4.28)], but they have the topology of ③.

As we have demonstrated in Sect. 4.3.2, we can take into account the imaginary mass counter term via differentiation, and the left-hand side of the unitarity equation (2.3) for ① + ② reads



(4.27)

The first term can never become resonant because it violates the cut structure and the second term is non-resonant as long as the amplitude itself is non-resonant which is true by our assumption $s \not\approx M^2$. Deriving the leading behavior of $\Delta^{\pm}(p,\mu)$ for small Γ/M for $p_0 \neq \pm\sqrt{\mathbf{p}^2 + M^2}$ we obtain

$$\begin{aligned}\Delta^{\pm}(p,\mu) &= \pm\mathrm{i}\frac{\left(-p_0 \pm 2\sqrt{\mathbf{p}^2+M^2}\right)}{2\sqrt{\mathbf{p}^2+M^2}^3\left(p_0 \mp \sqrt{\mathbf{p}^2+M^2}\right)^2}\Gamma M \\ &\quad + \mathcal{O}\left(\left(\frac{\Gamma}{M}\right)^3\right) \\ &= \pm\Gamma M f(\Gamma M),\end{aligned} \tag{4.28}$$

where f is a smooth function with Taylor expansion in ΓM. The explicit factor in ΓM indicates the suppression and after carrying out the differentiation in (4.27) the leading term of order ΓM is eliminated and the resulting order is $g^2 \times \mathcal{O}(\Gamma^2) = g^6$. Since g^4 is the current accuracy of the amplitude (4.25) the contributions from ① + ② are negligible, which is no coincidence.

The argument is easily extended to arbitrary high order. Consider an amplitude up to the order of g^n, where n is arbitrary. We compute the LTE according to (4.19) and assume we have a term $\mathcal{U} \in \mathcal{F}^{\tau}$ of the order of g^m and $m \leq n$ either violating the cut structure or having at least one non-resonant $\Delta^{\pm}(p,\mu)$. The order of $\mathcal{U}$ is bounded from below as follows:

$$\begin{aligned}\mathcal{O}(\mathcal{U}) &\geq g^m \sum_{k=0}^{\frac{n-m}{2}} \frac{1}{k!}\left(-\xi\frac{\partial}{\partial\Gamma M}\right)^k \Gamma M f(\Gamma M)\Bigg|_{\xi=\Gamma M} \\ &= \mathcal{O}\left(g^m \Gamma^{\frac{n-m}{2}+1}\right) = \mathcal{O}(g^{n+2}),\end{aligned} \tag{4.29}$$

where the equality (=) occurs solely for one non-resonant $\Delta^{\pm}$ in $\mathcal{U}$. Multiple insertions of $\mathrm{i}(-\mathrm{i}\Gamma M)$ result in a systematic elimination of orders as can be easily seen realizing that the differential operator in (4.19) is nothing but the Fourier representation of the translation operator

$$\left(\mathrm{e}^{-\xi\frac{\partial}{\partial\Gamma M}}\right)_{\frac{n-m}{2}} := \sum_{k=0}^{\frac{n-m}{2}} \frac{1}{k!}\left(-\xi\frac{\partial}{\partial\Gamma M}\right)^k, \tag{4.30}$$

where the series of the exponential function is terminated at the order $(\Gamma M)^{\frac{n-m}{2}}$. On the other hand, the translation operator acts as follows on a function P: $\mathrm{e}^{-\epsilon_0\frac{\partial}{\partial\epsilon}}P(\epsilon)\Big|_{\epsilon_0=\epsilon} = P(0)$. Thus, we obtain the same result with possible deviations starting at the order $(\Gamma M)^{\frac{n-m}{2}+1} = \mathcal{O}(g^{n-m+2})$ which shows the result (4.29).

Loosely speaking, the non-resonant propagators are expanded in Γ and thereafter the resummed and non-resummed terms explicitly cancel. As finite-width terms in the complex-mass counter term $\delta\mu^2$ result only from a reparametrization of the theory, resummed and non-resummed terms have to compensate each other in each fully calculated order.

The results so far can be summarized as follows:

- There exists a decomposition for the CMS propagator (4.11) satisfying the decomposition theorem (2.5), thus, allowing to derive a LTE [step ① in (4.6)].
- The LTE does not allow one to include the imaginary mass counter term directly, but it can be introduced via (4.19).
- We have shown that cut-structure-violating terms as well as all cuts of non-resonant propagators can always be neglected no matter at which order in the coupling constant the violation takes place. Thus, only correctly cut LTE amplitudes have to be taken into account, which is required by unitarity [step ② in (4.6)].

Further, from the stated results it follows immediately that unitarity is fulfilled automatically if there are no resonant $\Delta^{\pm}(p,\mu)$. The missing piece which has yet to be investigated is when $\Delta^{\pm}(p,\mu)$ becomes resonant. This happens when internal momenta are integrated out, or usually when certain phase-space integrations are carried out.

4.3.4 Resonant contributions of unstable particle propagators at one-loop order

In this section we discuss resonant $\Delta^{\pm}(p,\mu)$ at leading order in Γ/M. Those terms are no longer negligible in the LTE as for instance the second term of (4.15) for an unstable s-channel particle and $s \approx M^2$.

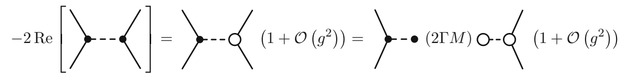
Naively interpreting the unitarity equation (2.3) would lead us to the conclusion that only the sums of all diagrams on both sides of the unitarity equation coincide, though, in the case of stable particles diagrams can be separated according to their topology and perturbative order. Perturbative unitarity then follows from the fact that the coupling can be chosen arbitrarily meaning that we can, in principle, distinguish between orders by varying the coupling. This argument cannot be directly transferred when the theory is renormalized according to the CMS. The distinction of perturbative orders does no longer work because of resummation, and we actually have to consider sums of diagrams, but the occurrence of non-trivial relations between topologically different Feynman diagrams can be excluded at least in scalar theories. As discussed in Sect. 4.1, we can consider relative orders in the sense that if the left-hand side of the unitarity equation has a phase-space-dependent order then also the right-hand side does, and the difference of orders is independent of phase space. Our strategy is therefore to identify the diagrams not only by their topology, but also by the perturbative order in certain phase-space regions where it is well defined, e.g. for $|s-M^2| \sim \Gamma M$ or $|s-M^2| \gg \Gamma M$.

At this point we recall that besides the loop expansion we only expand cut propagators $\Delta^{\pm}$ connecting the regions $\mathcal{T}, \mathcal{T}^*$ in Γ/M, but do never expand the propagators Δ within $\mathcal{T}$ or $\mathcal{T}^*$.

We start again with the example of the s-channel production of an unstable particle. For the resonant case one must perform a Laurent expansion to capture the leading behavior,

$$\Delta^{\pm}(p,\mu)\big|_{p_0=\pm\sqrt{\mathbf{p}^2+M^2}} = \mp\mathrm{i}\frac{2}{\Gamma M} + \mathcal{O}\left(\frac{\Gamma}{M}\right). \tag{4.31}$$

The LTE at LO reads for $p^2 = M^2$



(4.32)

where in the last step we made use of (4.31) and the identity

$$\frac{1}{\Gamma M} = \Delta(p,\mu)\Gamma M \Delta^*(p,\mu)\big|_{p^2=M^2}. \tag{4.33}$$

The LTE does not know about the diagrammatic significance of ΓM which we have to determine and plug in by hand, and, as discussed before, ΓM must be computed at least by the one-loop renormalization conditions (3.5). We denote Σ^1_{R} the renormalized (according to the CMS) one-loop self-energy of the unstable particle and Σ^1 the corresponding unrenormalized one. The self-energies are related to the counter term $\delta\mu^2$ by the renormalization conditions (3.5), and the consistency equation (3.4) links $\delta\mu^2$ and ΓM, so we obtain a relation between ΓM and Σ^1

$$\Gamma M = \mathrm{Im}[\delta\mu^2] = -\,\mathrm{Re}\left[\mathrm{i}\Sigma^1(p^2)\Big|_{p^2=\mu^2}\right]. \tag{4.34}$$

In the next step we assume that the analytic continuation of the self-energy behaves well enough at $p^2 \approx M^2$, i.e. we suppose that

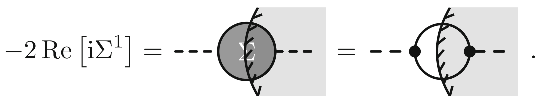
$$\left.\frac{\mathrm{i}\Sigma_{\mathrm{R}}(p^2)}{\Gamma M}\right|_{p^2\approx M^2} = \mathcal{O}(g^2), \tag{4.35}$$

which can be obtained formally by performing an expansion in p^2, but sometimes a Taylor expansion is not possible as is the case when infrared singularities appear. Then one usually has logarithmic corrections, but they do not bother us as long as the limit $g \to 0$ exists. In the next step we make use of the assumption (4.35) and find for ΓM:

$$\begin{aligned}\mathcal{O}(g^4) &= \mathrm{Re}\left[\mathrm{i}\Sigma_{\mathrm{R}}|_{p^2\approx M^2}\right] = \mathrm{Re}\left[\mathrm{i}\Sigma|_{p^2\approx M^2}\right] + \Gamma M\\ \Rightarrow \Gamma M &= -\,\mathrm{Re}\left[\mathrm{i}\Sigma|_{p^2\approx M^2}\right] + \mathcal{O}(g^4)\\ &= -\,\mathrm{Re}\left[\mathrm{i}\Sigma|_{p^2\approx M^2}\right](1+\mathcal{O}(g^2)).\end{aligned} \tag{4.36}$$

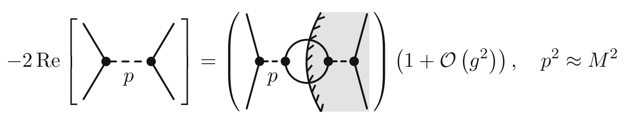
This equation expresses what is known from the on-shell scheme, i.e. the width is the cut through loops and can be interpreted as the decay width. At one-loop order the widths in the CMS and the on-shell scheme coincide, but this is no longer true at higher orders and we will not be able to argue this way in the general case.

Nevertheless, let us make use of this result to demonstrate unitarity at one-loop order. At this order we can directly apply the cutting rules to Σ^1 since there are no intermediate unstable particles (in our model)



(4.37)

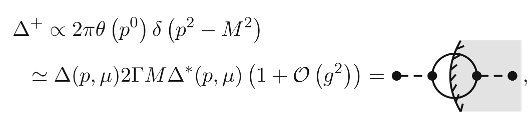
This result together with (4.32) and (4.34) yields exactly what is required by unitarity, namely



(4.38)

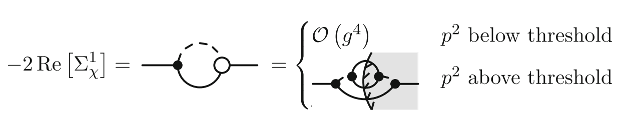
One can verify that no double counting occurs and that the one-loop amplitude ③ in (4.25) is canceled by the mass counter term ② up to terms of relative order $g^2 \sim \Gamma \sim p^2 - M^2$, as is required by the renormalization condition (3.5).

The Laurent expansion (4.31) is not appropriate when the momentum p is an internal loop or an integrated phase-space momentum. Instead we can make use of the solutions for the decomposition $\Delta^{\pm}(p,\mu)$ for small widths as given by the cutting rules (4.12) in Sect. 4.2. As Veltman has shown, one does not expect unstable particles in asymptotic states, and this dilemma is resolved at LO as follows. Similarly to the identity (4.33), we have in distributional sense that



(4.39)

where we used (4.34) and (4.37) in the last step. This shows that the LO resonant $\Delta^{\pm}$ can be interpreted as a higher-order cut amplitude which is what is required by Veltman's unitarity equation. For instance, consider the one-loop self-energy of the stable particle denoted as Σ^1_χ. Computing the LTE and making use of the result (4.39) yields up to higher orders



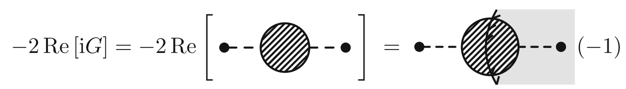
(4.40)

This can be generalized and proves (4.7) and (4.8) and thus unitarity as long as the leading behavior of $\Delta^{\pm}$ is sufficient.

4.3.5 Generalization to higher orders

In Sect. 4.3.4 we gave an interpretation for resonant $\Delta^{\pm}$ at leading order in Γ/M. In the following we generalize this result to arbitrary order by pursuing the strategy devised in steps ③ and ④ in (4.7) and (4.8).

For LTEs of higher-order amplitudes the approximation (4.39) is not sufficient, and more terms in the expansion of $\Delta^{\pm}$ in Γ/M must be taken into account. However, with the expansion the diagrammatic interpretation gets lost, and it is difficult to compare the result to the right-hand side of the unitarity equation. Motivated by Veltman's approach to derive LTEs for dressed propagators, we try to identify the LTE of higher-order two-point functions as higher-order cut two-point functions, i.e. we aim at defining cuts that fulfill

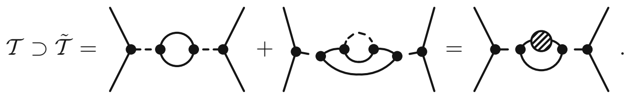


(4.41)

where $\mathrm{i}G$ represents the full propagator of the unstable particle. Provided that (4.41) is true, if the LTE of an arbitrary amplitude can be rearranged in such a way that the cut region is given by LTEs of two-point functions [step ③ in (4.7)] instead of the usual $\Delta^{\pm}$, then unitarity follows immediately since we have the correct cut structure (as shown in Sect. 4.3.3) and valid cuts (4.41) (to be defined). Thus, the problem is reduced to the problem of studying LTEs of two-point functions and this is necessary because the CMS mixes loop orders and only the LTE of the whole two-point function can yield well-defined cuts, which we show below.

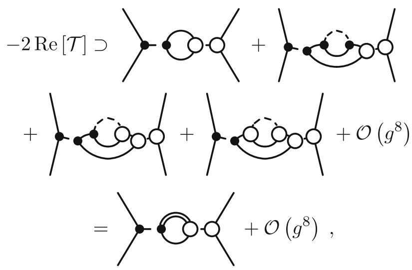
The idea is the following: For a specific amplitude we consider all diagrams up to a certain order in the CMS. Applying the LTE yields contributions with the correct cut structure (4.6). In contributions where unstable propagators are cut, these are iteratively replaced by cuts of the full propagator upon including the needed higher-order contributions [see (4.7)]. The validity of this replacement can be justified as follows. Consider a LTE contribution $\tilde{\mathcal{F}}$ with a $\Delta^{\pm}(p,\mu)$ originating from a specific CMS propagator $\Delta(p,\mu)$ somewhere in a diagram $\mathcal{F}$. If the diagram is of highest considered order, we can simply use (4.39) to replace $\Delta(p,\mu)$ by a cut through stable particles. If the diagram $\mathcal{F}$ is not of highest order, there are diagrams that have the same structure as $\mathcal{F}$ but more self-energy insertions next to that propagator. Among the contributions to the LTE of these higher-order diagrams are terms that have the same structure as $\tilde{\mathcal{F}}$, but where instead of the $\Delta^{\pm}$ LTE components of two-point functions appear (originating from the self-energy insertions), and collecting all these terms we retrieve $\tilde{\mathcal{F}}$ with a nested LTE of a two-point function.

We first give a simple example for a LTE of an amplitude with nested two-point functions. We show that the nested two-point functions reappear as a LTE of the two-point functions, i.e. we can identify cut two-point functions. Consider the subset of diagrams $\tilde{\mathcal{T}}$ of the complete $2 \to 2$ two-loop amplitude $\mathcal{T}$ defined by



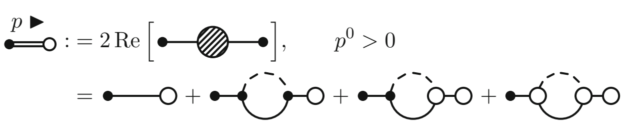
(4.42)

In this example, the hatched circle represents the propagator of the stable particle precisely up to one-loop order. For the purpose of demonstration, assume values of s off the resonance which is less complicated [the case $s \approx M^2$ is taken care of as described in (4.51) below]. Computing the LTE of $\mathcal{T}$, but only keeping the topologies of the kind of $\tilde{\mathcal{T}}$, yields



(4.43)

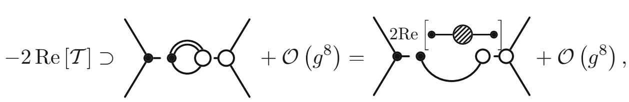
where



(4.44)

represents the LTE of the nested two-point function. Underlined end-points in two-point functions do not come with couplings and the underlining rules must be extended. We define the LTE of a two-point function by pretending there were couplings $(ig)^2$ at the end-points allowing us to make use of the usual underlining rules. After removing the end-point couplings (dividing by g^2), the difference between the two-point function with and without couplings is a sign, which is the reason why we have to take $2\,\mathrm{Re}$ instead of $-2\,\mathrm{Re}$.

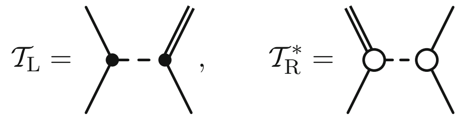
We arrive at



(4.45)

i.e. the cut through the two-point function defined as in (4.41) is expressed by a cut of a two-point function of lower order.

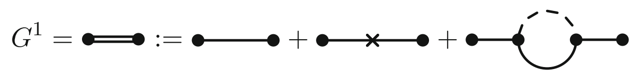
After isolating cut two-point functions the normal region $\mathcal{T}_\mathrm{L}$ and the complex-conjugated region $\mathcal{T}_\mathrm{R}^*$ are given by



(4.46)

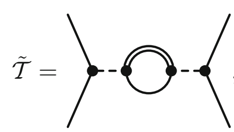
and notice that for this way of identifying terms in the LTE external states need not to be on-shell. The reformulation of the LTE (4.45) in terms of nested LTEs of two-point functions (4.44) is exactly step ③ in (4.7) which we elaborate now.

In our example (4.42) we derive a decomposition in momentum space for the one-loop propagator

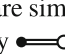
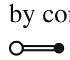


(4.47)

in terms of $G^{1,\pm}$, to be determined, which fulfill a space-time decomposition (2.5) where the Feynman propagator Δ_F is replaced by G^1. Such a decomposition of G^1 allows us to compute the LTE of amplitudes expressed in terms of G^1 propagators by the sum of all possible underlinings with suited expressions for $G^{1,+(-)}$ which must be the same for arbitrary amplitudes, but may be different for different propagators (different order or different particles). Further, the LTE diagrams must respect the cut structure since we have a physical situation and the computed sum of LTE amplitudes must equal the sum of LTE amplitudes obtained from the original amplitude without the identification of nested two-point functions. In order to derive $G^{1,+(-)}$ we start from the same amplitude (4.42),



(4.48)

where the doubly lined propagator is defined in (4.47) and computing the LTE, assuming we have a decomposition for (4.47), we obtain the result on the left-hand side of (4.45). The approximated solutions for the $G^{1,\pm}$ are simply read off by comparing with (4.43) and are given by  (4.44) and  the counter part where the energy flow is in the opposite direction which can be obtained by replacing black dots with white dots and vice versa in (4.44).

For a generic two-point function G the existence of an approximate decomposition is guaranteed by the fact that there is a decomposition for both stable and unstable (4.11) tree-level propagators and the perturbative cut structure (see Sect. 4.3.3), which is a consequence of kinematics. In general, the existence of a decomposition implies that the corresponding propagators must follow the underlining rules in

LTE diagrams and can only emerge as unchanged, complex conjugated or cut. Moreover, the two-point function behaves in the same way: It either stays the same, is complex conjugated or multiple terms can be associated to the cut region where the energy flow points in a specific direction. All other possible outcomes violate the cut structure. The solutions $G^{\pm}$ for a general two-point function are obtained by computing the LTE of G and making use of the cut structure [as in the example (4.44)]. The term $G^{+(-)}$ is given by the diagrams where the energy flow is to the right(left).

Returning to (4.7), instead of computing the LTE of an amplitude which is given by vertices and tree-level propagators, we can think of the same amplitude, but reformulated in terms of ●══●. In the LTE we encounter cut propagators like (4.44) which can be substituted by the real part of the two-point function, leading exactly to the right-hand side of (4.7).

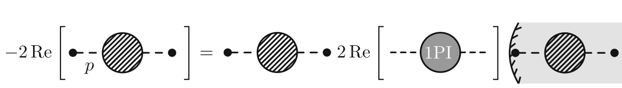
Having shown (4.7), i.e. how to rearrange LTEs of matrix elements in order to identify nested LTEs of two-point functions, we turn to the proof of (4.8) and elaborate on the meaning of (4.41). The diagrammatic significance of ΓM turns out to be a real problem beyond one loop in our current framework. At some point in our calculation, in particular when $\Delta^{\pm}$ is resonant, we have to plug in the expression for ΓM obtained from the renormalization condition. This problem can be circumvented by making use of resummed results, i.e. instead of using the usual perturbative expansion we represent two-point functions by their fully resummed equivalent deliberately taking into account non-significant (higher) perturbative orders. Then the partial resummation of the CMS is replaced by a complete resummation which turns out to be sufficient for a diagrammatic interpretation. Returning to the statement that, in contrast to the on-shell scheme, ΓM does not represent well-defined cut contributions, one realizes that

$$
\begin{aligned}
2\,\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}_{\mathrm{R}}(p^2)\right] &:= 2\,\mathrm{Re}\left[\mathrm{i}\Sigma_{\mathrm{R}}(p^2) - \Gamma M\right] \\
&= 2\,\mathrm{Re}\left[\mathrm{i}(\Sigma_{\mathrm{R}}(p^2) - \mu^2)\right]
\end{aligned}
\tag{4.49}
$$

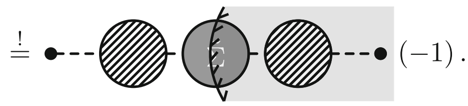
does, which can be understood as follows. We express the fully resummed two-point function as

$$
\begin{aligned}
\frac{\mathrm{i}}{p^2 - \mu^2 + \Sigma_{\mathrm{R}}} = \frac{\mathrm{i}}{p^2 - \mu^2 + \Sigma_{\mathrm{R}}} \left(\frac{p^2 - \mu^2 + \Sigma_{\mathrm{R}}}{\mathrm{i}} \right)^{*} \\
\times \left(\frac{\mathrm{i}}{p^2 - \mu^2 + \Sigma_{\mathrm{R}}} \right)^{*},
\end{aligned}
\tag{4.50}
$$

and computing the LTE of this expression we obtain

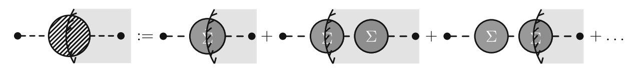


(4.51a)



(4.51b)

Note that when taking the real part of (4.50) one must only compute the real part of $\left[\mathrm{i}\left(p^2 - M^2\right) + \Gamma M - (\mathrm{i}\Sigma_{\mathrm{R}})^{*}\right]$ because the other factors form a real number. Further, the step (4.51a) is only allowed if $\mathrm{i}/(p^2 - \mu^2 + \Sigma_{\mathrm{R}})$ is non-singular which is true for unstable particles. Equation (4.51a) expresses the LTE of two-point functions by the LTE of self-energies. The equality with (4.51b) is equivalent to (4.41) allowing us to properly define cut two-point functions. Expanding the full propagators in (4.51b) in the CMS implies that the right-hand side of (4.41) should be defined as

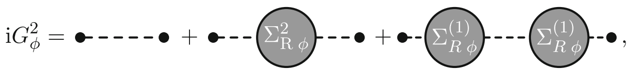


(4.52)

where the cut self-energies are defined by (4.49) and normal and complex-conjugated self-energies are just the usual ones renormalized according to the CMS. This definition of cuts is in agreement with Veltman's unitarity equation (4.3) as we show in the sequel.

In (4.49), (4.50), (4.51a) and (4.51b), i.e. for dressed propagators there is no partial resummation. Starting from the CMS, we can construct dressed propagators by resumming renormalized self-energies. After full resummation the $\mathrm{i}\Gamma M$ in the propagator cancels with the $\mathrm{i}\Gamma M$ of $\mathrm{i}\Sigma_{\mathrm{R}}$ and all explicit $\mathrm{i}\Gamma M$ expressions disappear which is the reason why (4.49) represents well-defined cuts. In this limit unitarity is not violated as has been shown by Veltman [27] and it is left to understand that nothing goes wrong when going from full resummation to the CMS. This is formally shown as follows: We only need to study cut two-point functions, i.e. we need to show that the right-hand side of (4.51a) is equal to (4.51b). Assume the left-hand side of (4.51a) is given at n-loop order, then (4.51a) tells us that the LTE of an n-loop two-point function can be computed by LTEs of n-loop self-energies, where we actually mean the self-energies (4.49). As in our example (4.45), these self-energy LTEs are iterated LTEs of one- to $(n-1)$-loop two-point functions. Thus, one makes the induction hypothesis that LTEs of two-point functions (4.51) represent well-defined cuts at $n-1$ loops. Expanding the two-point function on the left-hand side of (4.51a) at n loops the statement (4.51b) follows from the induction hypothesis, the start of the induction being given by (4.37). Thus, cuts are defined iteratively as in the example (4.45) and each cut through an unstable particle, possibly belonging to higher-order contributions and after collecting all terms like in (4.40) results in a nested cut two-point function (4.52) which itself can have cut unstable particles.

Let us illustrate the procedure at the example of our toy theory. Consider the two-point function of an unstable particle at two-loop order



(4.53)

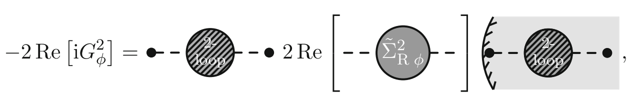
where $\Sigma^2_{\mathrm{R},\phi} = \Sigma^{(1)}_{\mathrm{R},\phi} + \Sigma^{(2)}_{\mathrm{R},\phi}$ is the two-loop self-energy of the unstable particle ϕ renormalized according to the CMS, and $\Sigma^{(1)}_{\mathrm{R},\phi}$ and $\Sigma^{(2)}_{\mathrm{R},\phi}$ denote the one-loop and two-loop renormalized contributions to the two-loop self-energy, respectively. At this point we have to require that the perturbation series is valid in the sense that the two-point function is well approximated by

$$\mathrm{i}G^2_\phi = \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\tilde{\Sigma}^2_{\mathrm{R},\phi}}\left(1 + \mathcal{O}\left(g^6\right)\right) =:$$

$$\left(1 + \mathcal{O}\left(g^6\right)\right),$$

(4.54)

where the two-point function on the right-hand side is defined as the resummed propagator for which the self-energy is evaluated and renormalized at two-loop order. Then we can compute the LTE with the help of (4.51a) and up to higher orders we have



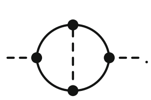
(4.55)

where $\tilde{\Sigma}^2_{\mathrm{R},\phi}$ is given by

$$\tilde{\Sigma}^2_{\mathrm{R},\phi} = \Sigma^{(2)}_\phi + \Sigma^{(1)}_\phi - \mathrm{Re}[\delta\mu^2]. \tag{4.56}$$

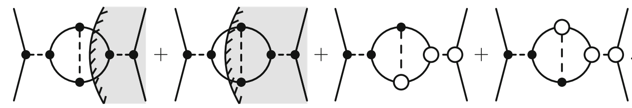
Notice that the imaginary part of $\delta\mu^2$ dropped out and only the real part is left over renormalizing the one- and two-loop self-energies. In what follows we could have argued with the induction hypothesis, but we work out this example explicitly and compute the LTE of $\Sigma^{(2)}_\phi$ and $\Sigma^{(1)}_\phi$. The LTE of $\Sigma^{(1)}_\phi$ is trivial and yields the one-loop cut (4.37). Among all contributions to $\Sigma^{(2)}_\phi$ there is none with nested unstable two-point functions except for the CMS propagator. For instance, consider the example

$$\Sigma^{(2)}_\phi \supset$$



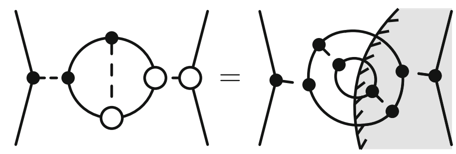
(4.57)

We apply the LTE and we only keep the terms with the correct cut structure since the other ones are negligible. Then we can compute the LTE for the two-point functions or matrix elements with the help of (4.55). Keeping only the terms directly related to our example (4.57), we obtain



(4.58)

As already mentioned, since there are no nested unstable two-point functions, we can directly apply (4.39)



$$\left(1 + \mathcal{O}\left(g^2\right)\right).$$

(4.59)

In this way we can show that all two-loop self-energies in this theory yield, up to higher orders, well-defined cut self-energies. Having dealt with the two-loop self-energies, if we combine the one- and two-loop result in (4.55), we have explicitly shown (4.51b) at two-loop order. In case of nested unstable two-point functions we would have had to use the induction hypothesis.

Finally, our results can be summarized as follows:

- In the beginning we demonstrated how to compute LTEs of arbitrary amplitudes leading us to the result that the cut structure is guaranteed in a perturbative sense, which is the basis for unitarity. It remains to check that $\Delta^\pm$ yields well-defined cuts.
- From the solutions of $\Delta^\pm$ for CMS propagators we concluded that unitarity holds off resonances. The leading behavior of resonant $\Delta^\pm$ can be interpreted as the cut one-loop two-point function [see (4.39)].
- The leading approximation of $\Delta^\pm$ is not enough beyond one-loop, and higher-order corrections need to be included. The LTEs of different loop orders do not separately represent valid cuts in the sense of Veltman's definition and must be considered simultaneously because owing to the partial resummation the diagrams are connected to each other and cancelations take place. Whenever there is a resonant unstable $\Delta^\pm$ we identify cuts by iteratively including appropriate higher-order contributions resulting in nested LTEs of two-point functions. For the LTE of the two-point functions up to a given order a valid cut interpretation can be assigned which is consistent with the interpretation of Veltman, i.e. only lines of stable particles are cut.

5 Conclusions

The CMS provides a straightforward method to consistently implement unstable particles in perturbative calculations.

Formally, the procedure is an analytic continuation of matrix elements to complex masses and (if necessary) couplings with appropriate renormalization condition.

In the CMS the Cutkosky cutting rules can no longer be used to verify unitarity, and it was not clear how perturbative unitarity is implemented. Following Veltman, we derived a largest-time equation within the CMS which could then be used to obtain a diagrammatic representation for the imaginary part of scattering amplitudes, also when unstable particles are present.

Our derivation of the largest-time equation is based on the decomposition theorem and we showed that an appropriate decomposition can be achieved for the CMS propagator. As a result, one finds that the would-be cut propagators $\Delta^{\pm}(p, \mu)$ of unstable particles are smoothed versions of the stable ones. In case of stable particles the Largest-Time Equation coincides with the Cutkosky cutting rules, but including unstable particles leads to additional contributions which can be interpreted as contributions where the energy flow is backward. Performing an expansion solely of would-be cut propagators $\Delta^{\pm}(p, \mu)$ of unstable particles in Γ/M does indeed yield cutting rules where unstable resonant $\Delta^{\pm}(p, \mu)$ can be replaced by higher-order cuts through stable particles only. In this way, we recover the perturbative statement of Veltman's result in the CMS, namely that a QFT is unitary up to higher orders only if unstable particles are excluded from asymptotic states. While we only considered a toy model with real couplings, the generalization to complex couplings is straightforward.

Funded by SCOAP^3.

References

1. R.G. Stuart, Phys. Lett. B **272**, 353–358 (1991)
2. A. Sirlin, Phys. Rev. Lett. **67**, 2127 (1991)
3. A. Sirlin, Phys. Lett. B **267**, 240 (1991)
4. U. Baur, J.A.M. Vermaseren, D. Zeppenfeld, Nucl. Phys. B **375**, 344 (1992)
5. E.N. Argyres et al., Phys. Lett. B **358**, 339 (1995). arXiv:hep-ph/9507216
6. W. Beenakker et al., Nucl. Phys. B **500**, 255–298 (1997). arXiv:hep-ph/9612260
7. L.F. Abbott, Nucl. Phys. B **185**, 189 (1981)
8. L.F. Abbott, M.T. Grisaru, R.K. Schaefer, Nucl. Phys. B **229**, 372 (1983)
9. A. Denner, S. Dittmaier, G. Weiglein, Acta Phys. Polon. B **27**, 3645 (1996). arXiv:hep-ph/9609422
10. A. Denner, S. Dittmaier, Phys. Rev. D **54**, 4499 (1996). arXiv:hep-ph/9603341
11. J. Papavassiliou, A. Pilaftsis, Phys. Rev. Lett. **75**, 3060–3063 (1995). arXiv:hep-ph/9506417
12. R.G. Stuart, Phys. Lett. B **262**, 113 (1991)
13. A. Aeppli, G.J. van Oldenborgh, D. Wyler, Nucl. Phys. B **428**, 126 (1994). arXiv:hep-ph/9312212
14. S. Dittmaier, A. Huss, C. Schwinn, Nucl. Phys. B **885**, 318 (2014). arXiv:1403.3216 [hep-ph]
15. W. Beenakker, F.A. Berends, A.P. Chapovsky, Nucl. Phys. B **573**, 503–535 (2000). arXiv:hep-ph/9909472
16. D.B. Franzosi, F. Maltoni, C. Zhang, Phys. Rev. D **87**, 053015 (2013). arXiv:1211.4835 [hep-ph]
17. M. Beneke, A.P. Chapovsky, A. Signer, G. Zanderighi, Phys. Rev. Lett. **93**, 011602 (2004). arXiv:hep-ph/0312331
18. M. Beneke, A.P. Chapovsky, A. Signer, G. Zanderighi, Nucl. Phys. B **686**, 205 (2004). arXiv:hep-ph/0401002
19. M. Beneke, Y. Kiyo, K. Schuller, arXiv:1312.4791
20. A. Denner, S. Dittmaier, M. Roth, L.H. Wieders, Nucl. Phys. B **724**, 247 (2005). arXiv:hep-ph/0505042. **(Erratum-ibid. B 854, 504 (2012))**
21. A. Denner, S. Dittmaier, M. Roth, D. Wackeroth, Nucl. Phys. B **560**, 33–65 (1999). arXiv:hep-ph/9904472
22. A. Denner, S. Dittmaier, Nucl. Phys. Proc. Suppl. **160**, 22–26 (2006). arXiv:hep-ph/0605312
23. S. Actis, G. Passarino, Nucl. Phys. B **777**, 100 (2007). arXiv:hep-ph/0612124
24. S. Actis, G. Passarino, C. Sturm, S. Uccirati, Phys. Lett. B **669**, 62 (2008). arXiv:0809.1302 [hep-ph]
25. S. Actis, G. Passarino, C. Sturm, S. Uccirati, Nucl. Phys. B **811**, 182 (2009). arXiv:0809.3667 [hep-ph]
26. G. Passarino, C. Sturm, S. Uccirati, Nucl. Phys. B **834**, 77 (2010). arXiv:1001.3360 [hep-ph]
27. M.J.G. Veltman, Physica **29**, 186 (1963)
28. T. Bauer, J. Gegelia, G. Japaridze, S. Scherer, Int. J. Mod. Phys. A **27**, 1250178 (2012). arXiv:1211.1684 [hep-ph]
29. R.E. Cutkosky, J. Math. Phys. **1**, 429–433 (1960)
30. G. 't Hooft, M.J.G. Veltman, NATO Adv. Study Inst. Ser. B Phys. **4**, 177 (1974)
31. A. Denner, Fortsch. Phys. **41**, 307–420 (1993). arXiv:0709.1075 [hep-ph]
32. M. Passera, A. Sirlin, Phys. Rev. Lett. **77**, 4146 (1996). arXiv:hep-ph/9607253
33. P. Gambino, P.A. Grassi, Phys. Rev. D **62**, 076002 (2000). arXiv:hep-ph/9907254
34. P.A. Grassi, B.A. Kniehl, A. Sirlin, Phys. Rev. D **65**, 085001 (2002). arXiv:hep-ph/0109228
35. D.Y. Bardin, A. Leike, T. Riemann, M. Sachwitz, Phys. Lett. B **206**, 539 (1988)
36. H. Veltman, Zeitschrift für Physik C Particles Field **62**, 35–51 (1994)
37. G. 't Hooft, 50 Years of Yang–Mills Theory. World Scientific Publishing, Singapore (2005)